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Neutrino masses and mixing from hierarchy and symmetry

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Abstract

We construct a model that allows us to determine the three neutrino masses and the mass matrix directly from the experimental mass squared differences Δ_{atm} and Δ_{sol}, anticipating rational hierarchy $(\mu m_1/m_2 = m_2/m_3)$, $\mu \approx 1$, and S_3–S_2 symmetry for the mixing matrix. We find that both the mass ratios and mixing angles are dominated by a parameter Λ. For the mixing angles, $\Lambda = \sqrt{1/6} \approx 0.41$, is a Clebsch–Gordan coefficient. For the masses, the mass ratios depend on the experimental Δ_{atm} and Δ_{sol} and with most recent data, remarkably, we also obtain $\sqrt{m_1/m_2} = \sqrt{m_2/m_3} = 0.41 = \Lambda$. This possibly coincidental equality gives a simple mass matrix in the $\sin(\theta_{13}) = 0$ limit. We find that with $\Delta_{\mathrm{sol}} = 8.2 \times 10^{-5}$ eV^2, $m_1 = 1.5 \times 10^{-3}$ eV, $m_2 = 9.2 \times 10^{-3}$ eV and $m_3 = 5.3(5.5) \times 10^{-2}$ eV for $\Delta_{\mathrm{atm}} = 2.73(2.95) \times 10^{-3}$ eV^2. We obtain the mass matrix M and evaluate it's elements numerically for the presently 'best fit' solution in the allowed range for $\sin(\theta_{13})$. We find that all matrix elements are smaller than 0.03 eV. The only candidates for double texture zeroes are M_{ee} and $M_{e\tau}$ or $M_{e\mu}$ (with $\theta_{13} \to -\theta_{13}$). The maximum effective mass for neutrinoless $\beta\beta$ decay is $|m_{\beta\beta}|_{\max} \approx 8 \times 10^{-3}$ eV.

1. Introduction

The twelve fermion masses of the Standard Model are, at present, arbitrary parameters. A grand unified model might, in principle, establish some relations among them. Although a very promising approach exists [1], there is no generally accepted model that establishes such relations. One of the things that we can do, in the mean time, is to look for empirical relationships or patterns. One such pattern, that of the 'rational' hierarchy of quarks and charged leptons, is well confirmed. By rational we mean that mass ratios of members of a family are very close to powers of a parameter λ [2]. For example, $m_b : m_s : m_d \approx 1 : \lambda^2 : \lambda^4$. This parameter also dominates the symmetry

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breaking exhibited by the mixing angles of the unitary matrix, which gives the flavor states as linear combinations of the mass eigenstates.

Mass patterns for neutrinos appear to be quite different from those of the charged fermions. The information for neutrinos comes mainly from solar and atmospheric neutrino oscillations [3,4]:

$$\Delta_{\rm sol} = \left|m^2_{\nu 2} - m^2_{\nu 1}\right| \approx 8.2^{+0.6}_{-0.5} \times 10^{-5}\ {\rm eV}^2 \quad \text{and} \quad \Delta_{\rm atm} = \left|m^2_{\nu 3} - m^2_{\nu 2}\right| \approx 2.73^{+0.8}_{-1.0} \times 10^{-3}\ {\rm eV}^2,$$

each at 90% CL.

In the following we determine the neutrino masses by proposing that, they too, follow a rational hierarchy and we determine the mass matrix by imposing S_3–S_2 symmetry. We observe a new relation between the mixing angles and the mass ratios. The mixing angle θ_{13} is small. In the limit θ_{13} goes to zero, we impose S_3–S_2 symmetry on the mixing matrix to fix the remaining mixing angles θ_{23} and θ_{12}. In a rational hierarchical model $m_2 \approx \sqrt{\Delta_{\rm sol}}$ and $m_3 \approx \sqrt{\Delta_{\rm atm} + \Delta_{\rm sol}}$. It then follows that m_1 must be small compared to $\Delta_{\rm sol}$. Motivated in part by the observed numerical similarity of $s_{12}s_{23}$ and $(\Delta_{\rm sol}/\Delta_{\rm atm})^{(1/4)}$, we equate the Cabibbo angle, $\sqrt{m_1/m_2} = \sqrt{m_2/m_3}$, to $s_{12}s_{23} = s_{12}c_{23}$, which will be named Λ, similar in spirit, but not in magnitude to the Wolfenstein parameter, λ. With this identification, the masses and the mass matrix are totally determined by $\Delta_{\rm sol}$ and $\Delta_{\rm atm}$.

2. Symmetry and hierarchy lead to a proposed new parameter for neutrino mass determination

The flavor states ν_e, ν_μ and ν_τ are related to the mass eigenstates ν_1, ν_2 and ν_3 by the unitary transformation U

$$U = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{13} & s_{13}e^{-i\partial} \\ s_{12}c_{23} + c_{12}s_{13}s_{23}e^{i\partial} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\partial} & -c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\partial} & c_{12}s_{23} + s_{12}s_{13}c_{23}e^{i\partial} & c_{13}c_{23} \end{pmatrix}. \tag{1}$$

There are two 'large' angles θ_{21} and θ_{23}. Setting the small angle θ_{13}, for which there is as yet no lower limit, equal to zero, we obtain U_0:

$$U_0 = \begin{pmatrix} c_{12} & -s_{12} & 0 \\ s_{12}c_{23} & c_{12}c_{23} & -s_{23} \\ s_{12}s_{23} & c_{12}s_{23} & c_{23} \end{pmatrix}. \tag{2}$$

The three columns of U are the three eigenvectors of the mass matrix in the $\theta_{13} = 0$ limit. If V_i is the ith column of U $(i = 1, 2, 3)$, then the mass matrix M is given by

$$M = \sum_i m_i V_i V_i^T, \tag{3}$$

where m_i is the ith eigenvalue of M.

It was proposed more than 15 years ago that the 'mass gap' of the hierarchical pattern is associated with pairing forces in analogy with Cooper pairs in BCS theory and the mass matrix of the neutral pseudoscalar mesons [5]. In this limit, the mass matrix is 'democratic' [6] and when diagonalized gives rise to only one massive state, the coherent state. The 'democratic' vector V_d is of particular interest here, where

$$V_d = \sqrt{(1/3)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4}$$

and

$$V_d V_d^\dagger = (1/3) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{5}$$

the 'democratic' matrix. The vector V_d was assigned to the heaviest mass, m_3, with pairing forces creating the mass gap in mind. The masses m_2 and m_1 were thought to be generated through a breaking of this S_3 symmetry, $S_3 \to S_2 \to S_1$ [5,7].

However, the smallness (or vanishing) of θ_{13} makes the BCS type mass gap for m_3 untenable in the neutrino case. In contrast to the BCS case, because of the vanishing of U_{e3} in U_0 (Eq. (2)), m_3 is a coherent mixture of $m_{\nu\mu}$ and $m_{\nu\tau}$, if $\theta_{23} = \pi/4$. Thus, we now have S_2 symmetry for m_3 and reserve S_3 symmetry for m_2. U_0 is now completely determined. This assignment of V_d as the eigenvector for m_2 has lately received considerable attention in the literature [8].

If rational hierarchy is to be our pattern, then we see from the definitions of Δ_{sol} and Δ_{atm}, that if $m_1 \ll \Delta_{\text{sol}}$ then the mass ratios $\sqrt{m_1/m_2} \approx \sqrt{m_2/m_3} \approx (\Delta_{\text{sol}}/\Delta_{\text{atm}})^{(1/4)}$ are implied. In fact, the 90% confidence limits on Δ_{sol} and Δ_{atm} with present data, imply that $0.39 \leqslant \sqrt{m_1/m_2} \approx \sqrt{m_2/m_3} \leqslant 0.46$. For the 'best fit' we obtain $\sqrt{m_1/m_2} \approx \sqrt{m_2/m_3} \approx 0.41$. Considering that from S3 symmetry we have $s_{12}s_{23} = \Lambda = \sqrt{1/6} = 0.41$, we suggest that this rough equality is not a coincidence.

We now relate the second large mixing angle, θ_{12}, to the mass ratio m_1/m_2 by the relation:

$$-s_{12}s_{23} \equiv \Lambda = \sqrt{m_1/m_2}. \tag{6}$$

This association of the mixing angles with the mass ratios was suggested by us earlier on phenomenological grounds [7], because both $s_{12}s_{23}$ and $(\Delta_{\text{sol}}/\Delta_{\text{atm}})^{(1/4)}$ are about the same, approximately equal to 0.4. We propose it here as a 'natural' pattern.

Considering s_{12} a small parameter for the moment (it is not), we get to first order in $s_{12}(c_{12} = 1)$ the matrix u_0:

$$u_0 = \begin{pmatrix} 1 & \sqrt{2}\Lambda & 0 \\ -\Lambda & \sqrt{1/2} & -\sqrt{1/2} \\ -\Lambda & \sqrt{1/2} & \sqrt{1/2} \end{pmatrix}, \tag{7a}$$

or more suggestively:

$$u_0 = \begin{pmatrix} 1 & \Lambda & \Lambda \\ -\Lambda & 1 & 0 \\ -\Lambda & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \\ 0 & \sqrt{1/2} & \sqrt{1/2} \end{pmatrix}. \tag{7b}$$

This shows the dynamic role assigned to θ_{12} by the assumption (6) and why we may consider it as 'natural'.

Restoring c_{12} and full unitarity we have for U_0:

$$U_0 = \begin{pmatrix} \sqrt{(1-2\Lambda^2)} & \sqrt{2}\Lambda & 0 \\ -\Lambda & \sqrt{1/2}\sqrt{(1-2\Lambda^2)} & -\sqrt{1/2} \\ -\Lambda & \sqrt{1/2}\sqrt{(1-2\Lambda^2)} & \sqrt{1/2} \end{pmatrix}. \tag{8}$$

Imposing S_3 symmetry (democracy) for the vector V_2 implies $U_{e2} = U_{\mu 2} = U_{\tau 2}$ or $\sqrt{2}\Lambda = \sqrt{1/2}\sqrt{1-2\Lambda^2}$, so that

$$U_0 = \begin{pmatrix} 2\Lambda & \sqrt{2}\Lambda & 0 \\ -\Lambda & \sqrt{2}\Lambda & -\sqrt{1/2} \\ -\Lambda & \sqrt{2}\Lambda & \sqrt{1/2} \end{pmatrix}. \tag{9}$$

By normalization, it follows that

$$-s_{12}s_{23} = \Lambda = \sqrt{1/6}. \tag{10}$$

Of course $\sqrt{1/6}$ is not a capricious number, inasmuch as along with $\sqrt{1/2}$ it is a Clebsch–Gordan coefficient [5,7]. What is a capricious notion is that it also is numerically equal to $\sqrt{m_1/m_2}$, which follows from the rational hierarchy, $\mu\sqrt{m_1/m_2} = \sqrt{m_2/m_3}$, with $\mu \approx 1$, as shown below.

For the mass ratios, we have

$$m_1/m_2 = m_1\Big/\sqrt{\Delta_{\rm sol} + m_1^2} \tag{11a}$$

and

$$m_2/m_3 = \sqrt{\Delta_{\rm sol} + m_1^2}\Big/\sqrt{\Delta_{\rm atm} + \Delta_{\rm sol} + m_1^2}. \tag{11b}$$

Setting $m_1/m_2 = m_2/m_3$, and solving for m_1 using present data ($\Delta_{\rm sol} = 8.2 \times 10^{-5}$ eV^2 and $\Delta_{\rm atm} = 2.75 \times 10^{-3}$ eV^2), we get $m_1 = 1.5 \times 10^{-3}$ eV and $\sqrt{m_1/m_2} = \sqrt{m_2/m_3} = (\Delta_{\rm sol}/\Delta_{\rm atm})^{(1/4)} = 0.41 = \sqrt{1/6}$.

It is, of course entirely possible that it is a coincidence that $s_{12}s_{23} \approx (\Delta_{\rm sol}/\Delta_{\rm atm})^{(1/4)}$ and that both are approximately equal to $\sqrt{1/6}$, which is the value demanded by S_3–S_2 symmetry, but we make this equality the basis of the present model. Hence Eq. (10).

We now have $-\sin(\theta_{23}) = \cos(\theta_{23}) = \sqrt{1/2}$ and $-\sin(\theta_{12}) = \sqrt{1/3} = \sqrt{2}\Lambda$, so that $\tan^2(\theta_{23}) = 1$ and $\tan^2(\theta_{12}) = 1/2$.

The hierarchy indicated here is not very strong, $m_2 \approx \Lambda^2 m_3 = (1/6)m_3$, so Λ should not be used as an expansion parameter. In fact, the situation is very different from the quark sectors. There, the possible S_3–S_2 symmetry is presumably the same for the d and u sectors and does not appear in the $V_{\rm ckm} = U_d^\dagger U_u$, which is then just 1. Only the symmetry breaking terms, dominated by powers of $\lambda \approx 0.23$ are seen and the underlying symmetry, if it exists, is obscured in the resulting Wolfenstein representation. The mixing angles can be large or small, depending on the assumed flavor basis. In the present model, on the other hand, Λ is intrinsic to the symmetry and must be $\sqrt{1/6} \approx 0.41$. An even weaker hierarchy has been proposed by Xing [9], where U and M are 'expanded' in terms of $\Lambda = U_{\mu 3} \approx \sin(\theta_{23}) \approx 0.7 \approx \sqrt{m_2/m_3}$.

3. The neutrino mass spectrum

Assuming the normal ordering of masses, $m_1^2 < m_2^2 < m_3^2$, we have two equations, for m_2^2 and m_3^2 in terms of the experimentally observed mass squared differences, $\Delta_{\rm sol}$ and $\Delta_{\rm atm}$.

$$m_2^2 = \Delta_{\rm sol} + m_1^2, \tag{12}$$

$$m_3^2 = \Delta_{\rm atm} + \Delta_{\rm sol} + m_1^2. \tag{13}$$

A mass scale is provided by a third equation, which relates m_1 and m_2 (see discussion above),

$$\sqrt{m_1/m_2} = \sqrt{1/6} = \Lambda. \tag{14}$$

Without loss of generality, but with an eye towards 'rational' hierarchy, we now represent the masses m_1, m_2, m_3 in terms of parameters Λ, m_3 and μ, where μ, measures the deviation from rational hierarchy

$$M_{\rm diag} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \sqrt{\Delta_{\rm sol} + m_1^2} & 0 \\ 0 & 0 & \sqrt{\Delta_{\rm atm} + \Delta_{\rm sol} + m_1^2} \end{pmatrix} = m_3 \begin{pmatrix} \mu\Lambda^4 & 0 & 0 \\ 0 & \mu\Lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Thus $m_2/m_3 = \mu\Lambda^2$ and $m_1/m_2 = \Lambda^2$. 'Perfect' rational hierarchy would mean $\mu = 1$ [2]. The data for $\Delta_{\rm sol}$ and $\Delta_{\rm atm}$ considerably restrict the possible solutions.

In Fig. 1 we display

$$\Delta_{\rm atm} = \Delta_{\rm sol}\left(\frac{\Lambda^4\mu^2 - 1}{\Lambda^4\mu^2(\Lambda^4 - 1)}\right), \tag{16}$$

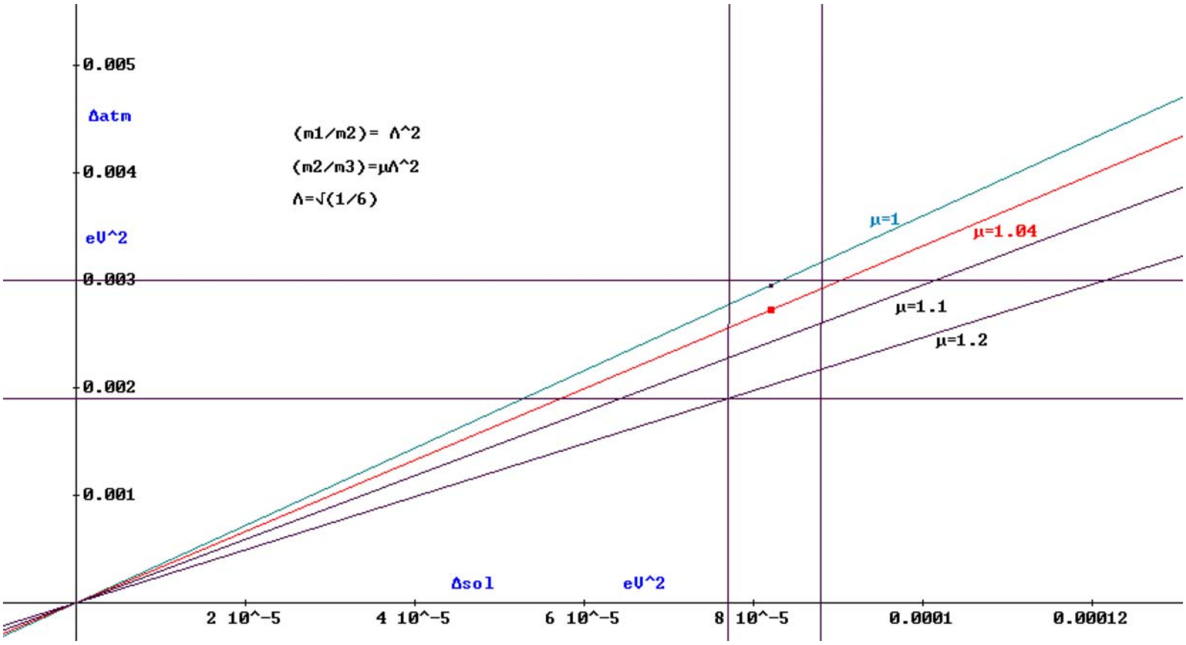

Fig. 1. $\Delta_{\rm atm}$ vs. $\Delta_{\rm sol}$ (Eq. (16) with $\Lambda^2 = 1/6$) for various values of μ ($\mu m_1/m_2 = m_2/m_3 = \mu\Lambda^2$). Acceptable solutions are within the (slightly arbitrary) rectangle $1.9 \times 10^{-3}\ \mathrm{eV}^2 < \Delta_{\rm atm} < 3 \times 10^{-3}\ \mathrm{eV}^2$ and $7.7 \times 10^{-5}\ \mathrm{eV}^2 < \Delta_{\rm sol} < 8.8 \times 10^{-5}\ \mathrm{eV}^2$. Two solutions are marked. They correspond to $\Delta_{\rm sol} = 8.2 \times 10^{-5}\ \mathrm{eV}^2$, with $\mu = 1$ (rational hierarchy) and $\mu = 1.04$ (best fit) [4].

which follows from the representation (15), to see the range, if any, of solutions consistent with the experimental range of $\Delta_{\rm sol}$ and $\Delta_{\rm atm}$. For clarity of the figure we chose a rectangle slightly smaller than the 2σ limits of Araki, et al., [3], and of Nakaya [4],

$$7.7 \times 10^{-5} < \Delta_{\rm sol} < 8.8 \times 10^{-5}\ \mathrm{eV}^2 \quad \text{and} \quad 1.9 \times 10^{-3} < \Delta_{\rm atm} < 3.0 \times 10^{-3}\ \mathrm{eV}^2.$$

With $\Delta_{\rm sol}$ and $\Delta_{\rm atm}$ given, all mass values are fixed. Since μ was totally unrestricted, it is gratifying to have $\mu \approx 1$ to be demanded by the acceptable data range, because values of μ much different from unity depart from the spirit of rational hierarchy. It is clear from Fig. 1 that μ ranges from 0.9–1.2 with $\Lambda = \sqrt{m_1/m_2} = \sqrt{1/6}$. For the 'best fit' values, $\Delta_{\rm sol} = 8.2 \times 10^{-5}\ \mathrm{eV}^2$ and $\Delta_{\rm atm} = 2.73 \times 10^{-3}\ \mathrm{eV}^2$, we have $\mu = 1.04$. This demonstrates the consistency of the model (Eq. (6)) with the notion of rational hierarchy and the oscillation data.

Eq. (15) can be solved to give the masses and the ratio parameter μ ($\mu m_1/m_2 = m_2/m_3 = \mu\Lambda^2$), as functions of $\Delta_{\rm atm}$ and $\Delta_{\rm sol}$

$$m_1^2 = \Delta_{\rm sol}\frac{\Lambda^4}{(1-\Lambda^4)}, \tag{17a}$$

$$m_2^2 = \Delta_{\rm sol}\frac{1}{(1-\Lambda^4)}, \tag{17b}$$

$$m_3^2 = \Delta_{\rm atm} + \Delta_{\rm sol}\frac{1}{(1-\Lambda^4)}, \tag{17c}$$

$$\mu^2 = \Delta_{\rm sol}\frac{1}{\Lambda^4(\Delta_{\rm atm}(1-\Lambda^4)+\Delta_{\rm sol})}. \tag{17d}$$

Eqs. (17) are valid independent of the choice of Λ. Substituting $\Lambda = \sqrt{1/6}$ from Eq. (10), we look at the properties of the two marked solutions of Fig. 1. Both have $\Delta_{\rm sol} = 8.2 \times 10^{-5}\ {\rm eV}^2$ (best fit) and differ only in that for the first solution we chose $\mu = 1$ (rational hierarchy) and let Eq. (16) determine $\Delta_{\rm atm}$, while for the second, we take $\Delta_{\rm atm} = 2.73 \times 10^{-3}\ {\rm eV}^2$ (best fit) and let μ be determined by Eq. (17d).

Since m_1 and m_2 depend only on Λ and $\Delta_{\rm sol}$, these masses are the same for both solutions; $m_1 = 1.53 \times 10^{-3}$ eV and $m_2 = 9.18 \times 10^{-3}$ eV:

Solution-1: 'rational hierarchy'. Input, $\mu = 1$: $\Delta_{\rm atm} = 2.95 \times 10^{-3}\ {\rm eV}^2$, $m_3 = 5.51 \times 10^{-2}$ eV,

Solution-2: 'best fit'. Input, $\Delta_{\rm atm} = 2.73 \times 10^{-3}\ {\rm eV}^2$: $\mu = 1.04$, $m_3 = 5.3 \times 10^{-2}$ eV.

Both solutions have the property $m_2 = 6m_1$ and $\mu m_3 = 6m_2$ with $\mu = 1$ and 1.04, respectively. $\Delta_{\rm atm} = |m_3^2 - m_2^2|$ and $\Delta_{\rm sol} = |m_2^2 - m_1^2|$ are within the acceptable experimental limits. All masses listed are absolute values.

4. Elements of the mass matrix and their properties

The mass matrix M is given by

$$M = U M_d U^T, \tag{18}$$

where U is given by (1) and

$$M_d = \begin{pmatrix} m_1 e^{i\alpha_1} & 0 & 0 \\ 0 & m_2 e^{i\alpha_2} & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{19}$$

The phases α_1 and α_2 are the Majorana phases. To get the matrix elements of M, listed below, we take α_1 to be $0, \alpha_2$ to be π and δ is the CP violating Dirac phase.

In the absence of symmetry breaking terms, $\sin(\theta_{13}) = 0$, we obtain the simple mass matrix:

$$M = m_3 \begin{pmatrix} 4\Lambda^6 - 2\Lambda^4 & -2\Lambda^6 - 2\Lambda^4 & -2\Lambda^6 - 2\Lambda^4 \\ -2\Lambda^6 - 2\Lambda^4 & \Lambda^6 - 2\Lambda^4 + \frac{1}{2} & \Lambda^6 - 2\Lambda^4 - \frac{1}{2} \\ -2\Lambda^6 - 2\Lambda^4 & \Lambda^6 - 2\Lambda^4 - \frac{1}{2} & \Lambda^6 - 2\Lambda^4 + \frac{1}{2} \end{pmatrix}, \tag{20}$$

where

$$\Lambda = \sqrt{1/6} \quad \text{and} \quad m_3 = \sqrt{\left(\Delta_{\rm atm} + \Delta_{\rm sol}\frac{1}{(1-\Lambda^4)}\right)} = 5.5 \times 10^{-3}\ {\rm eV}.$$

The elements of M are then given by

$$M_{ee} = 1/3(2m_1 - m_2) + s_{13}^2\big[1/3(m_1 + m_2) - m_1 + e^{-2i\delta} m_3\big], \tag{21a}$$

$$M_{e\mu} = M_{\mu e} = 1/\sqrt{2}c_{13}\big[-\sqrt{2}/3(m_1 + m_2) + s_{13}\big\{e^{i\delta}\big[m_1 - 1/3(m_1 + m_2)\big] - e^{-i\delta} m_3\big\}\big], \tag{21b}$$

$$M_{e\tau} = M_{\tau e} = 1/\sqrt{2}c_{13}\big[-\sqrt{2}/3(m_1 + m_2) - s_{13}\big\{e^{i\delta}\big[m_1 - 1/3(m_1 + m_2)\big] - e^{-i\delta} m_3\big\}\big], \tag{21c}$$

$$M_{\mu\mu} = 1/6(m_1 - 2m_2 + 3m_3) - 1/6 s_{13}\big[e^{i\delta} 2\sqrt{2}(m_1 + m_2) - s_{13}\big\{e^{2i\delta}(2m_1 - m_2) - 3m_3\big\}\big], \tag{21d}$$

$$M_{\mu\tau} = M_{\tau\mu} = 1/6(m_1 - 2m_2 - 3m_3) - 1/6\big[s_{13}^2\big\{e^{2i\delta}(2m_1 - m_2) - 3m_3\big\}\big], \tag{21e}$$

$$M_{\tau\tau} = 1/6(m_1 - 2m_2 + 3m_3) - 1/6 s_{13}\big[-e^{i\delta}2\sqrt{2}(m_1 + m_2) - s_{13}\big\{e^{2i\delta}(2m_1 - m_2) - 3m_3\big\}\big]. \tag{21f}$$

Fig. 2 shows the elements of M for solution-2 as functions of $\sin(\theta_{13})$ with $\delta = 0$. The maximum allowed $\sin(\theta_{13}) \approx 0.25$.

As may be seen from Fig. 2, the only candidates for double texture zeroes [10] are M_{ee} and $M_{e\tau}$ or $M_{e\mu}$ (with $\theta_{13} \to -\theta_{13}$). A double texture zero could be obtained with a moderate change in $\Delta_{\rm sol}$ and $\Delta_{\rm atm}$ [11], but not within their current experimentally acceptable limits. In addition, rational hierarchy would be badly violated. Consequently, we do not pursue this subject further.

The phase, δ, of the mixing matrix, U, has a serious effect for the mass matrix for the matrix elements $M_{e\mu}$ and $M_{e\tau}$, because for these elements the real part vanishes in the allowed range for θ_{13}, $\sin\theta_{13} \leqslant 0.25$ [12].

Fig. 3 shows $|M_{e\tau}|$ vs. $\sin(\theta_{13})$ for various values of δ. The values for $|M_{e\tau}|$ and $|M_{\tau e}|$ are the same as those for $|M_{e\mu}|$ and $|M_{\mu e}|$, with $\theta_{13} \to -\theta_{13}$.

The effective mass for neutrinoless $\beta\beta$ decay is

$$|m_{\beta\beta}| = \big|(2/3)c_{13}^2 m_1 e^{i\varphi_1} + (1/3)c_{13}^2 m_2 e^{i\varphi_2} + s_{13}^2 m_3\big|, \tag{22}$$

where $\varphi_{1,2} = \alpha_{1,2} + 2\delta$. To obtain $|m_{\beta\beta}|_{\max}$ we set φ_1 and $\varphi_2 = 0$. Using $m_1 = 1.5 \times 10^{-3}$ eV, $m_2 = 9.2 \times 10^{-3}$ eV and $m_3 = 5.5 \times 10^{-2}$ eV, we obtain $|m_{\beta\beta}|_{\max} \approx 8 \times 10^{-3}$ eV.

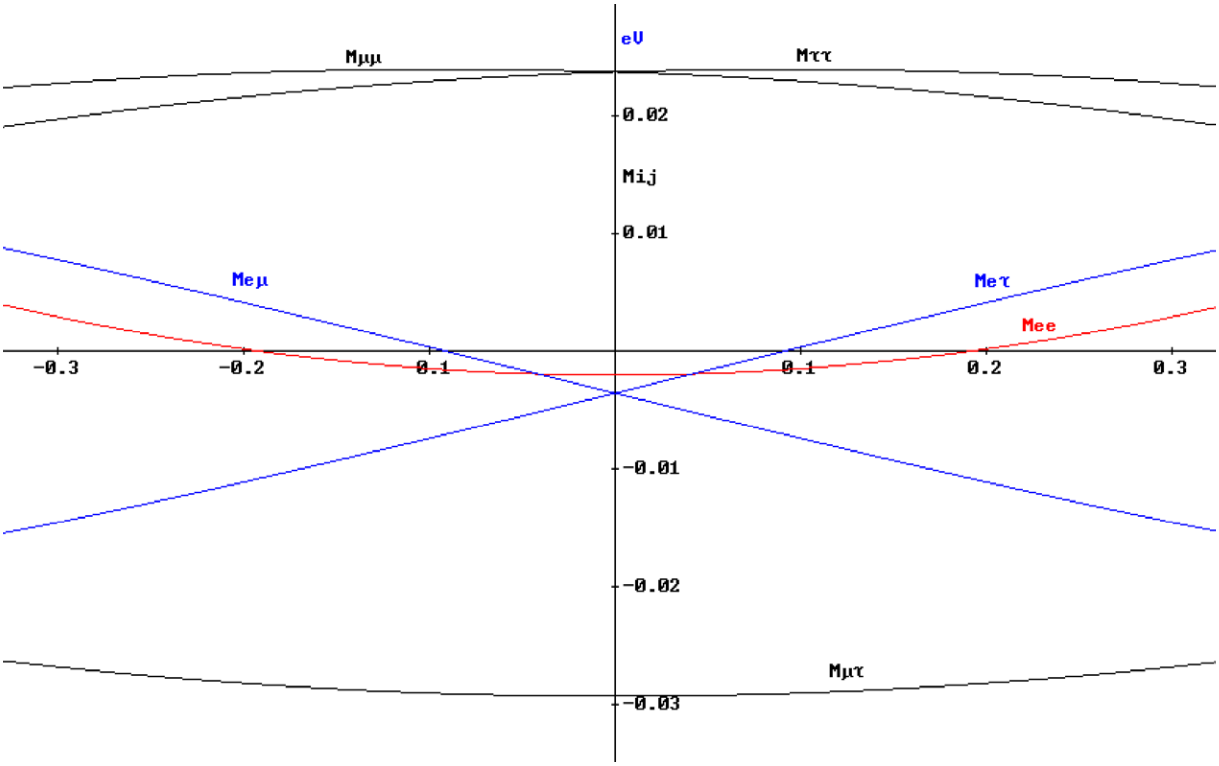

Fig. 2. Elements of the mass matrix M as functions of $\sin(\theta_{13})$, with $\delta = 0$. The masses are from solution-2, the 'best fit' solution, $\mu = 1.04$, $\Delta_{\rm sol} = 8.2 \times 10^{-5}$ eV^2, $\Delta_{\rm atm} = 2.73 \times 10^{-3}$ eV^2, and $m_2 e^{i\alpha_2} = -9.18 \times 10^{-3}$ eV. All elements are smaller than 0.03 eV.

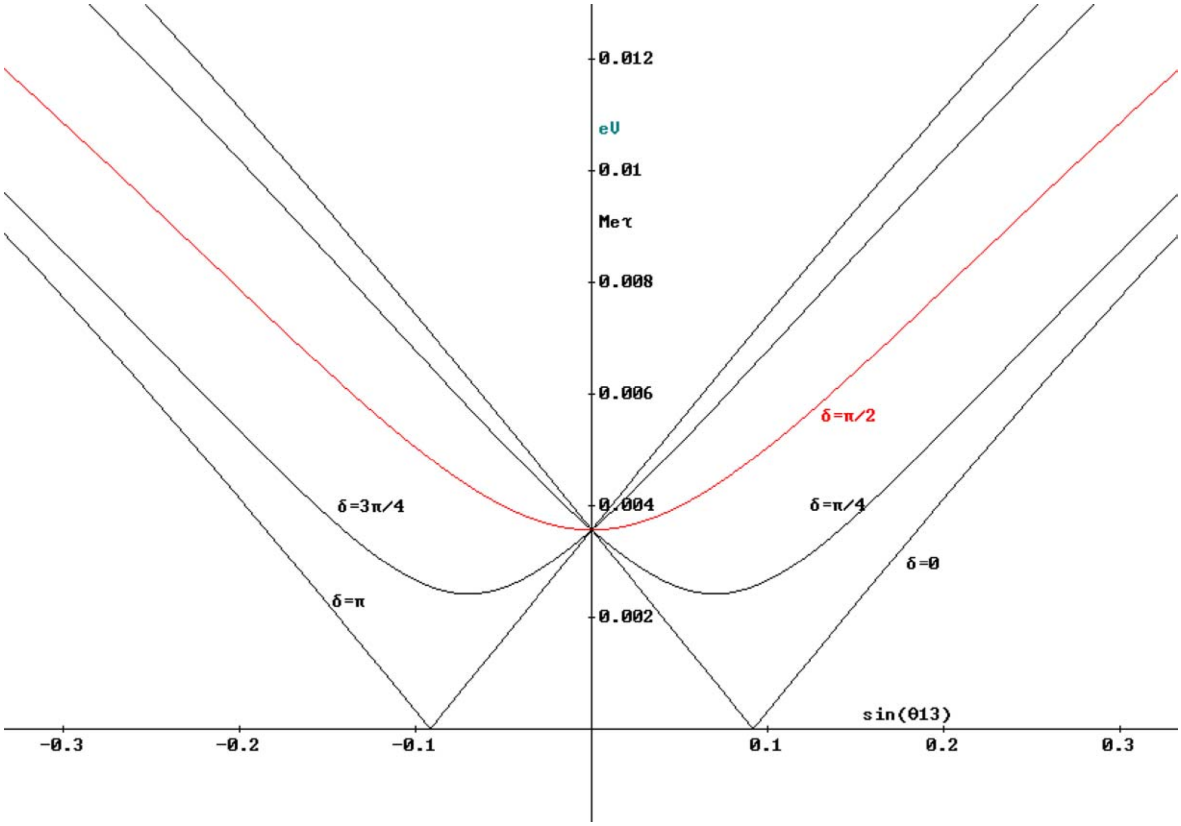

Fig. 3. $|M_{e\tau}|$ vs. $\sin(\theta_{13})$ for various values of δ, $0 \leqslant \delta \leqslant \pi$ in steps of $\pi/4$ for solution-2.

5. Conclusions

We have applied 'rational' hierarchy, i.e., $m_1 : m_2 : m_3 \approx \Lambda^4 : \Lambda^2 : 1$, to obtain the neutrino masses directly from the experimental mass squared differences, Δ_{atm} and Δ_{sol}. The mass matrix was formulated with the assumption of S_3–S_2 symmetry for the mixing matrix. Defining $-\sin(\theta_{12})\sin(\theta_{23}) = -\sin(\theta_{12})\cos(\theta_{23}) = \Lambda$, we find that Λ is the same both theoretically and derived from experimental data, i.e., $-s_{12}s_{23} = \sqrt{m_1/m_2} \equiv \Lambda = \sqrt{1/6}$. Consequently $m_1 \approx 1.5 \times 10^{-3}$ eV and $m_2 \approx 9.2 \times 10^{-3}$ eV. The largest mass, $m_3 \approx 5.3 \times 10^{-2}$ eV $\approx \sqrt{\Delta_{\text{atm}} + \Delta_{\text{sol}}}$. A study of the elements of the mass matrix, M, for our solution-2, that of the best fit solution, for the case $\delta = 0$, shows that all of them are smaller than 0.03 eV. The phase, δ, of the mixing matrix U has a serious effect for the mass matrix for the matrix elements $M_{e\mu}$ and $M_{e\tau}$, because for these elements the real part vanishes in the allowed range for θ_{13}, $\sin\theta_{13} \leqslant 0.25$. Their dependence on s_{13} for various values of δ is shown explicitly. We find that the maximum effective mass for neutrinoless $\beta\beta$ decay is, $|m_{\beta\beta}|_{\max} \approx 8 \times 10^{-3}$ eV.

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